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1 Introduction

Problem 1.1 (Minimum supports of eigenfunctions of graphs)**.** For a given graph G and its fixed eigenvalue λ find the minimum power of the support of an arbitrary λ-eigenfunction of the graph G.

The problem of minimal supports is closely related to the problem of intersection of combinatorial objects and the problem of finding the minimal size of combinatorial trades and null designs. In many cases, such problems can be considered as a special case of the minimum support problem for corresponding graphs with some discrete constraints on functions. For example, the classical results of T. Etzion and A. Vardi [EV94] on the intersection problem of perfect binary codes are directly related to the minimum support problem for the n-dimensional hypercube and eigenvalue -1; and the results of H. Huang's [H86]on trades combinatorial designs and S. Cho's [C98, C99] results on null designs in partially ordered sets are directly related to the minimum support problem for the smallest eigenvalue of Johnson and Grassmann graphs, respectively.

The problem of minimal supports was first formulated by D. C. Krotov and K. Vorobyov in [VK14]. In 2016, D. S. Krotov, I. Y. Mogilnykh and V. N. Potapov published an important paper [KMP16]. In this paper, they proved a lower bound on the size of the support of an arbitrary eigenfunction of a distantly regular graph (this bound is called the weight distribution bound - see Section 4 in the review of [SV21]). Moreover, in [KMP16] it was proved that for any distantly regular graph admitting a Delsarte pair, the existence of an eigenfunction which reaches the boundary of the weight distribution and corresponds to the smallest eigenvalue of the graph is equivalent to the existence of a regular bipartite subgraph of a special kind. After the publication of [KMP16] the minimal support problem was actively studied for various families of distantly regular graphs (see. [B18, GKSV18, S18, S19, V17, V21, VV19, VMV18]) and Cayley graphs on a symmetric group (see. [KKSV20]). In particular, the minimal support problem was completely solved for all eigenvalues of the Hamming graph (see. [K16, VV19, V21]). In addition, the minimal support problem was asymptotically solved for all eigenvalues of the Johnson graph in [VMV18].

In this paper, we consider the minimal support problem for the graph of a half cube. The paper is organized as follows. In Section 2, we give auxiliary definitions and a formulation of the problem. In Sections 4 and 5, we prove an estimate on the size of the minimal support of the eigenfunction for each eigenvalue. In Section 6, we give examples of eigenfunctions with minimal support for each eigenvalue.

2 Formulation, definitions

Let's $(\cdot, \cdot)$ - standard scalar product.

2.1 Graph of n-dimensional hypercube

A graph G = (V, E) is called an n-dimensional hypercube if
$V = \{0, 1\}^n$ и $E = \{\{u, v\} | u \in V, v \in V, (u - v, u - v) = 1\}$, that is 2 vertices are connected if they differ in exactly one position.

2.2 Graph of distances i

Let G = (V, E) be some graph. A graph $G_i = (V^{'}, E^{'})$ is called a Graph of distances i of graph G, if
$V^{'} = V$ and $E^{'} = \{\{u, v\} | u \in V, v \in V, d_G(u, v) = i\}$, where
$d_G : V \times V \rightarrow R_+$ - is a distance function in the graph G, i.e. $d_G(u, v)$ is the length of the shortest path between u, v in the graph G

2.3 Graph of n-dimensional half cube

A graph G = (V,E) is called a half-cube graph if
$V = \{v \in \{0, 1\}^n | \sum v_i = 0\ (mod 2)\}$ и $E = \{\{u, v\} | u \in V, v \in V, (u - v, u - v) = 2\}$

2.4 Eigenfunction of the graph

Let G(V, E) be a graph. The eigenfunction of the graph with eigenvalue λ, is the eigenvector of the adjacency matrix of the graph, with eigenvalue λ.
Let us define by N(v) - vertices belonging to V, which are connected to the vertex v.
The eigenfunction vector of f can be looked at as a function $f : V \rightarrow R$, moreover
$\forall v \in V, \sum_{u \in N(v)} f(u) = \lambda \cdot f(v)$

2.5 Eigenvalues of the hypercube

$\lambda_i = n - 2i, i \in [0...n]$

2.6 Eigenvalues of the half cube

$\theta_i = \frac{(n-2i)^2-n}{2}, i \in [0...[\frac{n}{2}]]$

2.7 Eigenvalues of the graph of distances 2 of the n-dimensional hypercube

Consider the graph of an n-dimensional hypercube, note that it is a bipartite graph whose components are vertices with even and odd sums of coordinates. Indeed, two vertices are connected if their coordinate entries differ exactly in one position, so the sums of their coordinates have different parity.

The graph of distance 2 of n-dimensional hypercube is a graph consisting of 2 connectivity components. These are the same components as the parts in the graph of the n-dimensional hypercube. Note that these components are isomorphic to the graph of the n-dimensional half cube.

So its eigenvalues coincide with the eigenvalues of the graph of the n-dimensional half cube.

2.8 Function support

The support of an arbitrary function $f : V \to R$ is the following set

$Support(f) := \{x \in V | f(x) \neq 0\}$

2.9 Formulation of the main problem

Finding the eigenfunction of a half-cube graph with minimal set of non-zero values for a fixed number of vertices and eigenvalue.

Let G(V, E) be a half cube on n vertices. λ - is an eigenvalue of G(V, E).

Find $min_{f \in U_\lambda, f \neq 0} |\{x \in V | f(x) \neq 0\}|$, where U_λ - is an eigenspace that corresponds to an eigenvalue of λ.

2.10 Basis of eigenspaces

Definition 2.1. Let's define the functions $\chi_u(x) := (-1)^{(u,x)}$, where $u \in Z_2^n$.

Claim 2.1. *The set* $\{\chi_u : u \in Z_2^n\}$ *forms an orthonormalized basis in the space of real-valued functions on the graph* G

Claim 2.2. *The set* $\{\chi_u : u \in Z_2^n, |u| = i\}$ *forms a basis in the eigenspace corresponding to the eigenvalue (n-2i) for the graph of an n-dimensional hypercube.*

2.10.1 Proof.

Definition 2.2. Let us define the scalar product on the space of real-valued functions defined on the set Z_2^n as follows.
$\langle f, g\rangle = \frac{1}{2^n}\sum_{x\in Z_2^n} f(x)g(x)$

Definition 2.3. A character of a finite group G is a function $\chi : G \to C^*$, such that $\forall g, h \in G, \chi(g \cdot h) = \chi(g) \cdot \chi(h)$

Note that the functions $\chi_u(x)$ are characters for the group Z_2^n.

Claim 2.3. *If* χ *is a nontrivial character of a finite group G, then*
$\sum_{g\in G}\chi(g) = 0$

Proof. Since χ is nontrivial, there exists g', such that $\chi(g') \neq 1$. Then
$\sum_{g\in G}\chi(g) = \sum_{g\in G}\chi(g \cdot g') = \sum_{g\in G}\chi(g)\chi(g')$. Hence, we get
$(\chi(g') - 1) \cdot \sum_{g\in G}\chi(g) = 0$ ☐

We can now prove claim 2.1.

Proof. 1) $\langle \chi_u, \chi_u\rangle = \frac{1}{2^n}\sum_{x\in Z_2^n}(-1)^{2(u,x)} = \frac{2^n}{2^n} = 1$
2) $\langle \chi_u, \chi_v\rangle = \frac{1}{2^n}\sum_{x\in Z_2^n}(-1)^{(u,x)}(-1)^{(v,x)} = \frac{1}{2^n}\sum_{x\in Z_2^n}(-1)^{(u+v,x)} = \frac{1}{2^n}\sum_{x\in Z_2^n}\chi_{u+v}(x) = 0$
So, this set is linearly independent and its cardinality is 2^n, so it is a basis since the dimension of the whole space is 2^n. ☐

Let's prove claim 2.2.
Let us fix u such that in u the first i coordinates are 1 and the remaining coordinates are 0. Consider a vertex v of the graph with all coordinates equal to 0. $\chi_u(v) = 1$, consider the neighbors of vertex v in the graph of an n-dimensional cube, these are vertices with exactly 1 num in their coordinate record. They can be divided into 2 parts.
1) w, such that the one in the last n-i coordinates, $\chi_u(w) = 1$, such n-i neighbors
2) w, such that the one in the first i coordinates, $\chi_u(w) = -1$, such neighbors i
By calculating the sum of the values over the neighbors, we obtain the eigenvalue n-2i. Similarly, all functions $\chi_u(x)$ are eigenfunctions. Moreover, the eigenvalue depends only on the number of units in u, hence we obtain that
$\{\chi_u(x) | u \in Z_2^n, (u, u) = i\}$ - form the basis of the eigenspace E_{λ_i}.

2.11 Relation of eigenspaces of hypercube and half-cube

Let us denote by V_{θ_i} the eigenspace for the graph of distance 2 of the n-dimensional hypercube, which corresponds to the eigenvalue $\theta_i = \frac{(n-2i)^2-n}{2}$,and by E_{λ_i} — the eigenspace for the hypercube graph, which corresponds to the eigenvalue $\lambda_i = n-2i$

Claim 2.4. $V_{\theta_i} = E_{\lambda_i} \oplus E_{\lambda_{n-i}} = E_{\lambda_i} \oplus E_{-\lambda_i}$.

Proof. Consider the basis vector $\chi_u(x)$ from the space $E_{\lambda i}$. Let us rearrange the coordinates so that in u the first i coordinates are 1 and the remaining coordinates are 0.
Consider a vertex v of the graph with all coordinates equal to 0.
$\chi_u(v) = 1$, consider the neighbors of vertex v in the graph of a half n-dimensional cube, these are vertices with exactly 2 ones in their coordinate record. They can be divided into 3 parts.
1) w such that 2 units in the last n-i coordinates, $\chi_u(w) = 1$, such neighbors are C^2_{n-2}
2) w, one by one among the first i and last n-i coordinates,
$\chi_u(w) = -1$, such neighbors i·(n-2)
3) w such that 2 ones in the first i coordinates, $\chi_u(w) = 1$,
such neighbors C_i^2

Let's calculate the sum of values of the function$\chi_u(x)$ over the neighbors.
$\sum_{w\in N(v)} \chi_u(w) = C^2_{n-2} - i(n-1) + C_i^2 \overset{!}{=} \frac{(n-2i)^2-n}{2}$
$(n-i)(n-i-1) - 2i(n-i) + i(i-1) \overset{!}{=} (n-2i)^2 - n$
Similarly, all basis vectors from $E_{-\lambda_i}$ also lie in $V_{\theta i}$. That is, we obtain that $\sum_i E_{\lambda i} \subset \sum_j V_{\theta j}$. Given that $\sum_i E_{\lambda i} = \sum_j V_{\theta j}$, пwe get that all embeddings are equal and as a result $V_{\theta_i} = E_{\lambda_i} \oplus E_{\lambda_{n-i}} = E_{\lambda_i} \oplus E_{-\lambda_i}$ □

3 Main result

Definition 3.1. Let $res(n, \theta_i)$ be the minimum power of the support of the eigenfunction of a half n-dimensional hypercube for eigenvalue θ_i.

The main result of the paper is the following theorem.

Theorem 3.1. *For all* $i \in [0...[\frac{n}{2}]]$ *the equality*

$$res(n, \theta_i) = 2^{\frac{n+\sqrt{n+2\theta_i}}{2}-1} = 2^{n-i-1}$$

4 Case of odd n

Claim 4.1 (The claim about symmetry of eigenfunctions of the hypercube)**.** *Let f be an eigenfunction of an n-dimensional hypercube with eigenvalue* λ_i*.*
Then $\forall x \in Z_2^n, f(x) = (-1)^i f(\mathbf{1} - x)$*, where **1** is a vector with all coordinates equal to 1*

Proof. Consider an eigenfunction from the basis $E_{\lambda i}$ $\chi_u(x)$ for the vertex u, whose first i coordinates are 1 and the rest are 0, and prove equality for it.
Consider a vertex x. Let x have a one-coordinates among the first i coordinates. Then really $\chi_u(x) = (-1)^a = (-1)^i \cdot (-1)^{i-a} = (-1)^i \cdot \chi_u(\mathbf{1} - x)$ □

4.1 A simple consequence for the case of odd n

Let f be an eigenfunction of the graph of distance 2 of an n-dimensional hypercube for an eigenvalue θ_i with minimal support capacity
Note that it is identically equal to 0 on one of the connectivity components. According to the claim that $V_{\theta_i} = E_{\lambda_i} \oplus E_{-\lambda_i}$, there exist functions $g \in E_{\lambda_i}, h \in E_{-\lambda_i}$ such that $f = g + h$.
Then the functions g and h have opposite values on one of the components.
Without begging for generality, let us say that this is the component of vertices with even sums of coordinates.

Claim 4.2 (Relationship of functions g and h)**.**

Proof. Let us prove that g, h have opposite values on vertices with even sum of coordinates and the same values on other vertices.
Indeed, since the graph of an n-dimensional hypercube is bipartite, determining the values of the eigenfunction on one of the parts will uniquely set the values on the vertices from the other part.
Let V_{odd} be vertices with odd sum of coordinates and V_{even} with even.
Consider the vertex $u \in V_{odd}$.
$g(u) = \frac{1}{\lambda_i} \cdot \sum_{v \in N(u)} g(v) = \frac{1}{\lambda_i} \cdot \sum_{v \in N(u)} (-h(v)) = \frac{-1}{\lambda_i} \cdot \sum_{v \in N(u)} h(v) = h(u)$

□

Then the size of the minimal support f can be expressed from the size of the minimal support for the graph of an n-dimensional hypercube. Namely, as a minimum from the contraction of the support into sets with even and odd sum of coordinates.

More precisely: $min(|\{x \in V_{odd}|f(x) \neq 0\}|, |\{x \in V_{even}|f(x) \neq 0\}|)$

Now, it matters for us what fraction of the n-dimensional hypercube contains less zeros of the function g. But since $\forall x \in Z_2^n, f(x) = (-1)^n f(\mathbf{1} - x)$,

and for odd n vertices x and **1**-x lie in different parts, then the number of zeros of the function g in parts with even and odd sum of coordinates is equal. Hence we get the answer.

5 Proof for any n

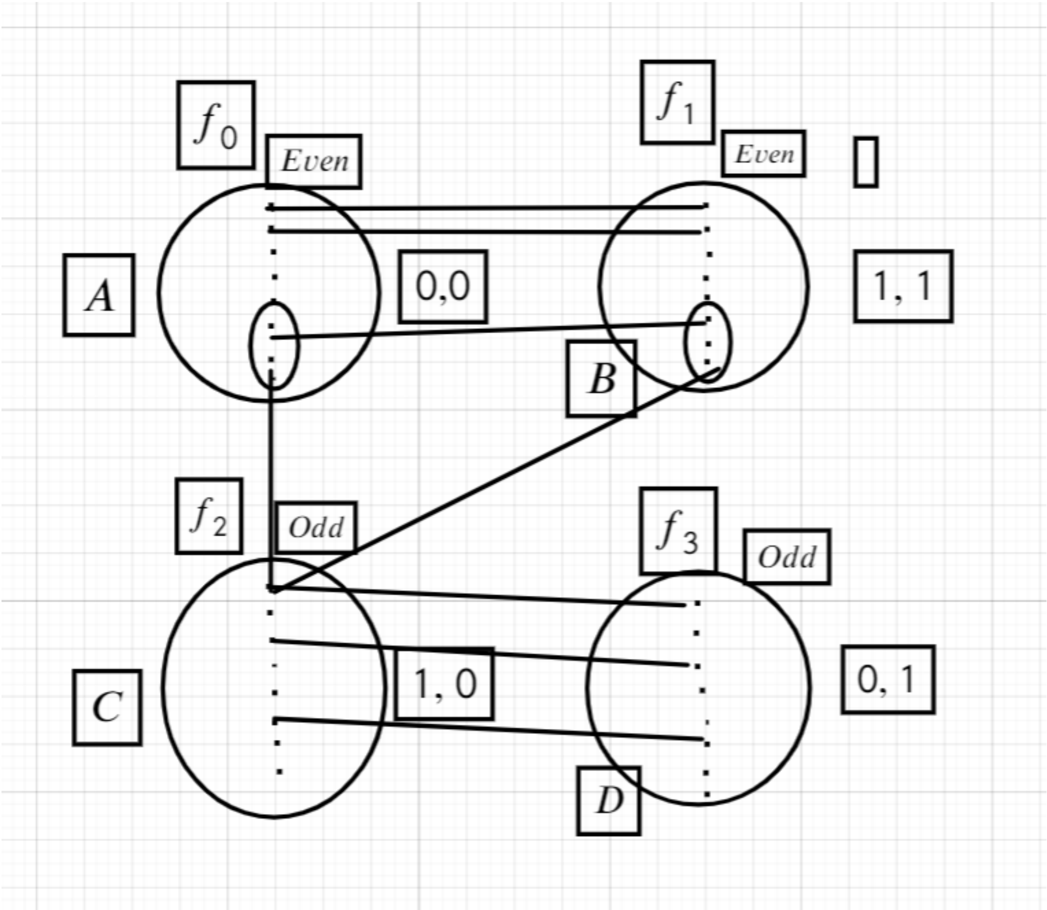
Consider the graph of an n-dimensional half hypercube, denote it by G(n).

Let f be an eigenfunction of this graph.

Let us distinguish 4 sets of vertices A, B, C, D.

These are vertices whose coordinate entries end at (0,0), (1,1), (1,0) and (0, 1), respectively.

Let us also introduce 4 new functions f_0, f_1, f_2, f_3. These are constraints of the function f on the sets A, B, C, D. That is, the domain of definition of these functions is A, B, C, and D, and the values coincide with the values of the function f on these sets.



Remark. *Note that the subgraphs of graph G(n) induced by sets A, B, C, D are isomorphic to graph G(n-2)*

We will prove an estimate on the minimum carrier size by induction.

5.1 Claim of induction

$res(n, \theta) = 2^{\frac{n+\sqrt{n+2\theta}}{2}-1}$

5.2 Base

For any natural n, $res(n, \frac{n^2-n}{2}) = 2^{n-1}$. Indeed, in this case the values of the function are identically equal on all vertices. So the support is the whole set of vertices.

5.3 transition

We'll do an induction transition $(n-2, \theta+1) \to (n, \theta)$
Suppose the opposite: we find an eigenfunction f for graph G(n) with eigenvalue θ_i and whose support capacity is less than $res(n, \theta_i)$
Idea: by the function f we construct a new eigenfunction for the graph
G(n-2), whose carrier is not much smaller than that of f.

Let's define a bijection b that translates vertices between sets (A, B) and (C, D). The vertices are mapped into each other if they differ only in the last 2 coordinates.
Consider the following eigenfunctions of the graph G(n)
$g(v) := f_1(b(v))$, if v $\in A$
$g(v) := f_0(b(v))$, if v $\in B$
$g(v) := f_2(v)$, if v $\in C$
$g(v) := f_3(v)$, if v $\in D$

$h(v) := f_3(b(v))$, if v $\in C$
$h(v) := f_2(b(v))$, if v $\in D$
$h(v) := f_1(v)$, if v $\in A$
$h(v) := f_2(v)$, if v $\in B$

Claim 5.1. *g is an eigenfunction of graph $G(n)$ with eigenvalue θ.*

Proof. Let's consider v $\in A$.
Note that the neighbors v and $b(v)$ in sets C and D are the same vertices. Indeed, if $x \in C, \{x, v\} \in E$, then x and v differ in exactly 2 positions in the coordinate notation. So they differ in exactly 1 position in the first n-2 coordinates. So the difference between x and $b(v)$ is also exactly in 2 coordinates, so they are connected. That is
$N(v) \cap (C \cup D) = N(b(v)) \cap (C \cup D)$.

Since the graphs G[A], G[B] are isomorphic and b is an isomorphism function.
$\sum_{u \in N(v) \cap A} g(u) = \sum_{u \in N(b(v)) \cap B} f(u)$, since $g(u) = f(b(u))$

Since the neighbors v and $b(v)$ in sets C and D are the same vertices. $\sum_{u \in N(v) \cap (C \cup D)} g(u) = \sum_{u \in N(b(v)) \cap (C \cup D)} f(u)$

From the definition of g. $g(b(v)) = f(v)$
Summing all together, we get
$\theta \cdot g(v) = \theta \cdot f_1(b(v)) = \theta \cdot f(b(v)) = \sum_{u \in N(b(v)) \cap B} f(u) + \sum_{u \in N(b(v)) \cap (C \cup D)} f(u) + f(v) = \sum_{u \in N(v) \cap A} g(u) + \sum_{u \in N(v) \cap (C \cup D)} g(u) + g(b(v))$

Similarly h is an eigenfunction of G(n) with eigenvalue θ

□

Corollary 5.1.1. *Since the functions f, g, h are eigenfunctions, the functions (f-g), (f-h) are also eigenfunctions.*

Claim 5.2. *(f-g), (f-h) are eigenvalues for G[A] and G[C] respectively with eigenvalues* $\theta + 1$.

Proof. We prove that the functions (f-g), (f-h) are eigenfunctions for G[A] and G[C] respectively with eigenvalues $\theta + 1$.
Indeed, (f-g) is an eigenfunction with eigenvalue θ
Let's $v \in A$.

Let us expand the definition of the eigenfunction for v, given that the values of (f-g) on C and D are identically equal to 0
$\theta \cdot (f-g)(v) = \sum_{u \in N(v) \cap A} (f-g)(u) + (f-g)(b(v))$
Note that. $(f-g)(v) = (-1)(f-g)(b(v))$
As a result, we obtain that (f-g) is an eigenfunction for G[A], with eigenvalue $\theta + 1$. ☐

Claim 5.3. *Estimation of the sizes of the supports of functions (f-g)*

Proof. Consider a vertex v such that $(f-g)(v) \neq 0$.
Then either $f(v) \neq 0$ or $g(v) \neq 0$
Hence we obtain the newline estimate
$|S(f-g)| \leq |S(f)| + |S(g)|$ ☐

Let us restrict the functions (f-g), (f-h) to A and C respectively. Using the previous lemma we obtain the following:
$|S((f-g)|_A)| = (S((f_0 - f_1(b))|_A)| \leq |S(f_0)| + |S(f_1)|$
$|S((f-h)|_C)| = (S((f_2 - f_3(b))|_A)| \leq |S(f_2)| + |S(f_3)|$
Учитывая что $|S(f)| = |S(f_0)| + |S(f_1)| + |S(f_2)| + |S(f_3)|$ without detracting from the generality, let's say that
$|S((f-g)|_A)| \leq \frac{|S(f)|}{2}$.
Initially, we assumed that $|S(f)| \leq res(n, \theta)$
That is, we have found an eigenfunction in a graph isomorphic to G(n-2) with eigenvalue $\theta + 1$, whose support is $\leq \frac{res(n,\theta)}{2}$

Claim 5.4. $res(n, \theta) = 2 \cdot res(n-2, \theta+1)$

Proof. $2^{\frac{(n-\sqrt{n+2\theta})}{2}+\sqrt{n+2\theta}-1} \overset{!}{=} 2 \cdot 2^{\frac{((n-2)-\sqrt{(n-2)+2(\theta+1)})}{2}+\sqrt{(n-2)+2(\theta+1)}-1}$
$2 \cdot 2^{\frac{((n-2)-\sqrt{(n-2)+2(\theta+1)})}{2}+\sqrt{(n-2)+2(\theta+1)}-1} = 2^{\frac{2+((n-2)-\sqrt{(n-2+2)+2(\theta)})}{2}+\sqrt{(n-2+2)+2(\theta)}-1}$ ☐

We got a contradiction, the transition is proven

6 The structure of the intrinsic function

Let G(n) be the graph of an n-dimensional hypercube.

Let the eigenfunction f with an eigenvalue θ be constructed for the graph of a half n-dimensional hypercube.

Let us construct an eigenfunction for the graph of an (n+2)-dimensional half hypercube with eigenvalue $\theta + 1$.

Similarly, let us define sets A, B, C, D in graph G(n+2).

Similarly, the subgraphs induced by sets A, B, C, D are isomorphic to G(n).

Let's define the bijection function b, b.

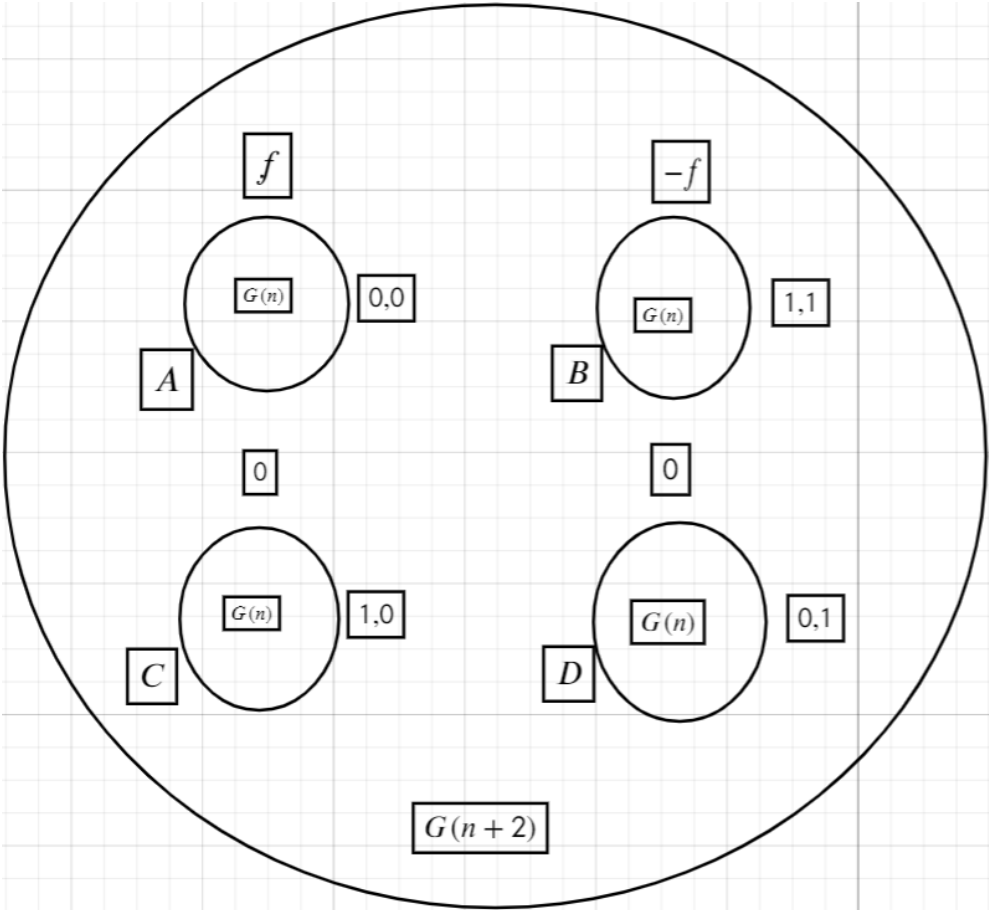
Set the values of the new function F.

$F(v) := f(v)$, if v $\in A$

$F(v) := -f(v)$, if v $\in B$

$F(v) := 0$, if v $\in C$

$F(v) := 0$, if v $\in D$



Claim 6.1. *F(v) is an eigenfunction of G(n+2) with eigenvalue* $\theta - 1$

Proof. Consider a vertex v, if $v \in C$ or $v \in D$, then

$\sum_{u\in N(v)} F(u) = \sum_{u\in N(v)\cap C} F(u) + F(b(v)) + \sum_{u\in N(v)\cap A} F(u) + \sum_{u\in N(v)\cap B} F(u) =$
$= \sum_{u\in N(v)\cap A} F(u) + \sum_{u\in N(v)\cap B} F(u) =$
$= \sum_{u\in N(v)\cap A} F(u) + \sum_{u\in N(v)\cap A} F(b(u)) = 0 = F(v)$
If $v \in A$, then by definitions $\theta \cdot F(v) = \sum_{u\in N(v)\cap A} F(u)$,
$F(b(v)) = -F(v)$.
Summing it up, we get the following $(\theta - 1)F(v) = \sum_{u\in N(v)} F(u)$ $\square$

That is, we have learned to construct eigenfunctions with the minimum possible support capacity for all eigenvalues of a half n-dimensional hypercube whose domain of values is $\{-1, 0, 1\}$

7 Conclusion

In the framework of this paper, in the general case for the graph of a half n-dimensional hypercube and all its eigenvalues, we prove estimates for the minimum possible support capacity and give examples of eigenfunctions for all eigenvalues on which optimal lower estimates are achieved.

The question of the general structure of the eigenfunctions of the half n-dimensional n-dimensional hypercube on which the lower estimate for the support size is achieved remains open.

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